Forces, Energy and Equations of Motion

Chapter 1

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Chem. 4300

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Galileo taught us how to measure time.

Pendulum’s period is approximately independent of swing amplitude.

In 1632 he published “Dialogue Concerning the Two Chief World Systems” where he proposed his law of inertia and introduces inertial frames.
In 1686 Issac Newton publishes “Principia” which gives his laws of motion, his law of universal gravitation, and his derivation of Kepler’s laws of planetary motion.
Newton’s Laws of Motion - The First Law

Newton’s First Law is the *Law of Inertia*.
In an inertial reference frame, a particle either remains at rest or continues to move at a constant velocity, $\vec{v}$, unless acted upon by a net external force.

**Definition**
Inertial frame: any reference frame in which Newton’s first law holds, that is, a non-accelerating, non-rotating frame.

Can you give examples of inertial and non-inertial frames?
Newton’s Laws of Motion - The Second Law

Newton’s Second Law

For any particle of mass, \( m \), the net force, \( \vec{F} \), on the particle is always equal to the mass, \( m \), times the particle’s acceleration, \( \vec{a} \),

\[
\vec{F} = m\vec{a}
\]

The 2\textsuperscript{nd} law is more generally given in terms of the particle’s momentum, \( \vec{p} = m\vec{v} \), as

\[
\vec{F} = \frac{d\vec{p}}{dt}
\]

A force acting on a particle can cause a change in the particle’s momentum vector (direction and magnitude).
Newton’s Laws of Motion

Consider a system of $N$ particles each of different mass $m_\alpha$. 

![Diagram of particles in 3D space with coordinates x, y, z and masses m1, m2, m3, m4, m5.]
Newton’s Laws of Motion

Consider a system of $N$ particles each of different mass $m_\alpha$.

Net force on $\alpha^{th}$ particle, $\vec{F}_{\text{net}}^\alpha$, is given by

$$\vec{F}_{\text{net}}^\alpha = \vec{F}_{\alpha}^{\text{ext}} + \sum_{\beta=1}^{N} \vec{f}_{\alpha\beta}$$

- $\vec{F}_{\alpha}^{\text{ext}}$ is external force acting on $\alpha^{th}$ particle.
- $\vec{f}_{\alpha\beta}$ is internal force on $\alpha^{th}$ particle by $\beta^{th}$ particle.
Newton’s Laws of Motion

Consider a system of \( N \) particles each of different mass \( m_\alpha \).
Total momentum, \( \vec{p}_{\text{total}} \), of system of particles is

\[
\vec{p}_{\text{total}} = \sum_{\alpha=1}^{N} \vec{p}_\alpha
\]

\( \vec{p}_\alpha \) is momentum of \( \alpha^{\text{th}} \) particle.

We write Newton’s 2nd law as

\[
\vec{F}_{\text{total}} = \frac{d\vec{p}_{\text{total}}}{dt} = \sum_{\alpha=1}^{N} \frac{d\vec{p}_\alpha}{dt} = \sum_{\alpha=1}^{N} \vec{F}_{\alpha}^{\text{net}} = \sum_{\alpha=1}^{N} \vec{F}_{\alpha}^{\text{ext}} + \sum_{\alpha=1}^{N} \sum_{\beta=1, \beta \neq \alpha}^{N} \vec{f}_{\alpha\beta}
\]
Newton’s Laws of Motion - The Third Law

Consider a system of $N$ particles each of different mass $m_\alpha$.

$$\vec{F}_{\text{total}} = \frac{d\vec{p}_{\text{total}}}{dt} = \sum_{\alpha=1}^{N} \vec{F}_{\alpha}^{\text{ext}} + \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \vec{f}_{\alpha\beta}$$

Newton’s third law

If object 1 exerts force $\vec{f}_{21}$ on object 2, then object 2 always exerts reaction force $\vec{f}_{12}$ on object 1

$$\vec{f}_{12} = -\vec{f}_{21}$$

Double summation term reduces to zero with 3rd law leaving

$$\vec{F}_{\text{total}} = \frac{d\vec{p}_{\text{total}}}{dt} = \sum_{\alpha=1}^{N} \vec{F}_{\alpha}^{\text{ext}} + \sum_{\alpha=1}^{N} \sum_{\beta=1, \beta \neq \alpha}^{N} \vec{f}_{\alpha\beta} = \sum_{\alpha=1}^{N} \vec{F}_{\alpha}^{\text{ext}} = \vec{F}_{\text{total}}^{\text{ext}}$$
Conservation of Linear Momentum

Consider a system of $N$ particles each of different mass $m_\alpha$.

$$\text{if } \frac{d\vec{p}_{\text{total}}}{dt} = 0, \text{ then } \vec{p}_{\text{total}} = \text{constant}$$

- When no net external force acts on system of particles, $\vec{F}_{\text{ext, total}} = 0$, then $\vec{p}_{\text{total}}$ is constant.

- This is the principle of conservation of linear momentum.

- It is true in classical and quantum mechanics.
Conservation of Linear Momentum

Example

A 100 kg astronaut was doing repairs outside her space ship when suddenly the onboard computer malfunctions and disconnects her tether line, leaving her out of reach of the airlock hatch door. In an attempt to return to the ship she throws a 3.0 kg wrench away from her at 5.0 m/s relative to the space station. With what speed and in which direction will she begin to move? Assume the astronaut and wrench are initially at rest relative to the space ship so that $\vec{v}_{\text{wrench}} = \vec{v}_{\text{astro}} = 0$. 

P. J. Grandinetti (Chem. 4300)
Conservation of Linear Momentum

Solution

Conservation of momentum says total momentum must remain unchanged before and after she throws wrench

\[ \vec{p}_{\text{total, initial}} = \vec{p}_{\text{total, final}} = \text{constant} \]

If she throws wrench in \( x \) direction we have

\[ m^{(\text{astro})} v^{(\text{astro})}_{x, \text{initial}} + m^{(\text{wrench})} v^{(\text{wrench})}_{x, \text{initial}} = m^{(\text{astro})} v^{(\text{astro})}_{x, \text{final}} + m^{(\text{wrench})} v^{(\text{wrench})}_{x, \text{final}} = \text{constant} \]

Taking initial velocities of \( v^{(\text{wrench})}_{\text{initial}} = v^{(\text{astro})}_{\text{initial}} = 0 \) leaves

\[ m^{(\text{astro})} v^{(\text{astro})}_{x, \text{final}} + m^{(\text{wrench})} v^{(\text{wrench})}_{x, \text{final}} = 0 \]

or

\[ v^{(\text{astro})}_{x, \text{final}} = -\frac{m^{(\text{wrench})}}{m^{(\text{astro})}} v^{(\text{wrench})}_{x, \text{final}} \]

She moves in opposite direction that wrench is thrown.
Center of Mass
Baton toss in strobe light
Center of Mass

Consider a system of $N$ particles each of different mass $m_\alpha$.

Equation of motion for $\alpha^{th}$ particle is given by

$$m_\alpha \ddot{r}_\alpha = \vec{F}_{\text{net}}^\alpha \quad \text{or} \quad m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \vec{F}_\alpha^{\text{ext}} + \sum_{\beta=1, \beta\neq \alpha}^{N} \vec{f}_\alpha\beta$$

Summing over all $N$ particles gives

$$\sum_{\alpha=1}^{N} m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \sum_{\alpha=1}^{N} \vec{F}_\alpha^{\text{ext}} + \sum_{\alpha=1}^{N} \sum_{\beta=1, \beta\neq \alpha}^{N} \vec{f}_\alpha\beta$$

$3^{rd}$ law makes double sum zero

Define total force acting on system of particles as

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_\alpha^{\text{ext}} \quad \text{to get} \quad \sum_{\alpha=1}^{N} m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \vec{F}$$
Center of Mass

Consider a system of $N$ particles each of different mass $m_\alpha$.

$$\sum_{\alpha=1}^{N} m_\alpha \frac{d^2 \vec{r}_\alpha}{dt^2} = \vec{F}$$

Defining

- the total mass as $M = \sum_{\alpha=1}^{N} m_\alpha$
- the center of mass as $\vec{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha$

we obtain

$$M \frac{d^2 \vec{R}}{dt^2} = \vec{F}$$

Center of mass obeys same equation of motion as single particle of total mass $M$ acted on by force, $F$.

Particles thrown into the air execute a complicated motion while center of mass follows a simple parabolic path like a single particle.
Baton toss in strobe light
Example

Find center of mass of water molecule. Take OH bond length as \( d = 0.957 \, \text{Å} \), H-O-H angle as \( \Omega = 105^\circ \), and mass of oxygen and hydrogen as 16.0 u and 1.0 u. (1 u = 1.66053904 \times 10^{-27} \, \text{kg})

- With mass of atom concentrated at nucleus we safely approximate each atom as point mass.
- Place oxygen atom at origin and hydrogen symmetrically in \( x-z \) plane above and below \( x \) axis.
Center of mass of water

Example

Find center of mass of water molecule. Take OH bond length as \( d = 0.957 \, \text{Å} \), H-O-H angle as \( \Omega = 105^\circ \), and mass of oxygen and hydrogen as 16.0 u and 1.0 u. 
(1 u = 1.66053904 \times 10^{-27} \, \text{kg})

- \( x \) coordinate of each hydrogen is \( d \cos \Omega/2 \)
- \( z \) coordinate is \( \pm d \sin \Omega/2 \) for H above and below \( x \) axis, respectively.

<table>
<thead>
<tr>
<th>#</th>
<th>atom</th>
<th>mass</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>H</td>
<td>1 u</td>
<td>+0.5826 Å</td>
<td>0</td>
<td>+0.7592 Å</td>
</tr>
<tr>
<td>2</td>
<td>H</td>
<td>1 u</td>
<td>+0.5826 Å</td>
<td>0</td>
<td>−0.7592 Å</td>
</tr>
<tr>
<td>3</td>
<td>O</td>
<td>16 u</td>
<td>0</td>
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Center of mass of water

Example

Find center of mass of water molecule. Take OH bond length as $d = 0.957$ Å, H-O-H angle as $\Omega = 105^\circ$, and mass of oxygen and hydrogen as 16.0 u and 1.0 u.

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Substituting values into formulas

$$X = \frac{m_Hx_1 + m_Hx_2 + m_Ox_3}{m_H + m_H + m_O} = \frac{(1.0 \text{ u})(0.5826\text{Å}) + (1.0 \text{ u})(0.5826\text{Å})}{1.0 \text{ u} + 1.0 \text{ u} + 16.0 \text{ u}} = 0.06473\text{Å}$$

$$Z = \frac{m_Hz_1 + m_Hz_2 + m_Oz_3}{m_H + m_H + m_O} = \frac{(1.0 \text{ u})(0.7592\text{Å}) + (1.0 \text{ u})(-0.7592\text{Å})}{1.0 \text{ u} + 1.0 \text{ u} + 16.0 \text{ u}} = 0$$
Center of mass of water

Example

Center of mass of water is along $x$ axis and shifted slightly to the oxygen position by $0.065$ Å (see small black dot).
Work, Power, and Kinetic Energy
Work and Power

Work

*Work* done by external force on particle moving it from $\vec{r}_1$ to $\vec{r}_2$ is given by *path integral*

$$w = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{\text{ext}} \cdot d\vec{r}$$

The SI unit for work is the joule ($1 \text{ J} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$).

Power

Work done per unit time is called *power*

$$P = \frac{dw}{dt} = \vec{F}_{\text{ext}} \cdot \frac{d\vec{r}}{dt} = \vec{F}_{\text{ext}} \cdot \vec{v}$$

The SI unit for power is the watt ($1 \text{ W} = 1 \text{ J/s} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^3$).
Work-Energy Theorem

\[ P = \frac{dw}{dt} = \vec{F}^{\text{ext}} \cdot \frac{d\vec{r}}{dt} = \vec{F}^{\text{ext}} \cdot \vec{v} \]

Substituting Newton’s 2\textsuperscript{nd} law, \( \vec{F}^{\text{ext}} = m\vec{a} = m\frac{d\vec{v}}{dt} \)

\[
\frac{dw}{dt} = m \frac{d\vec{v}}{dt} \cdot \vec{v} = m v_x \frac{dv_x}{dt} + m v_y \frac{dv_y}{dt} + m v_z \frac{dv_z}{dt}
\]

Note time dependence of \( \vec{v}(t) \) and rewrite the above expression as

\[
\frac{dw}{dt} = d \left( \frac{1}{2} m v_x^2(t) + \frac{1}{2} m v_y^2(t) + \frac{1}{2} m v_z^2(t) \right) = d \left( \frac{1}{2} m v^2(t) \right)
\]

Work done on the particle is transformed into kinetic energy

\[
w(t_1 \rightarrow t_2) = \Delta K = \frac{1}{2} m v^2(t_2) - \frac{1}{2} m v^2(t_1)
\]
Conservative Forces and Potential Energy

Conservative Force

*Conservative force* acting on particle depends only particle’s position, \( \vec{r} \), and not on particle’s velocity, \( \vec{v} \), time, \( t \), or any other variable.

When conservative force acts on particle traveling in closed loop the net work done on particle is zero, that is,

\[
w = \oint \vec{F} \cdot d\vec{r} = 0
\]

Potential Energy

Since work done by conservative force is same for all paths between points 1 and 2, we can write

\[
w(1 \to 2) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = -[V(\vec{r}_2) - V(\vec{r}_1)]
\]

and define \( V(\vec{r}) \) as a quantity called the *potential energy*. 

\( V(\vec{r}) \) is a *scalar field*: a physical quantity that can have a different value at each point in space and time.
Conservative Forces and the Work-Energy Theorem

When work-energy theorem involves conservative force it becomes

\[ w(1 \rightarrow 2) = K_2 - K_1 = -V(\vec{r}_2) + V(\vec{r}_1) \]

*Total mechanical energy, \( E \), in moving particle between \( \vec{r}_1 \) and \( \vec{r}_2 \) by conservative force is*

\[ E = K_1 + V(\vec{r}_1) = K_2 + V(\vec{r}_2) \]

Total mechanical energy is constant when forces acting on particle are conservative.
Conservative Forces and Potential Energy

When all forces acting on particle are conservative we calculate force vector from gradient of scalar potential energy,

$$\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r})$$

$\vec{\nabla}$ is gradient operator

$$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

When applied to potential energy function we obtain gradient vector of $V(x, y, z)$:

$$\vec{\nabla}V(x, y, z) = \vec{e}_x \frac{\partial V(x, y, z)}{\partial x} + \vec{e}_y \frac{\partial V(x, y, z)}{\partial y} + \vec{e}_z \frac{\partial V(x, y, z)}{\partial z}$$

$\vec{\nabla}V(x, y, z)$ is vector that points in direction of greatest increase of potential energy function.
Conservative Forces and Potential Energy

Potential Energy Surface in One Dimension

\[ F = -\frac{dV(x)}{dx} \]

\[ E = K + V(x) \]
Equations of Motion in terms of Potential Energy

For particle of mass $m$ acted upon by conservative forces Newton’s 2$^{\text{nd}}$ law becomes three coupled second-order partial differential equations:

$$ m \frac{d^2 x}{dt^2} + \frac{\partial V(x, y, z)}{\partial x} = 0 $$

$$ m \frac{d^2 y}{dt^2} + \frac{\partial V(x, y, z)}{\partial y} = 0 $$

$$ m \frac{d^2 z}{dt^2} + \frac{\partial V(x, y, z)}{\partial z} = 0 $$

Equations are coupled because potential energy function, $V(x, y, z)$, is present in every equation.
Classification of Differential Equations

Definition

Order of differential equation is order of highest derivative in equation.

Definition

Ordinary differential equation, \( F \left( t, f, \frac{df}{dx}, \ldots, \frac{d^nf}{dx^n} \right) = 0 \), is linear if \( F \) is linear function of variables \( f, \frac{dy}{dx}, \ldots, \frac{d^nf}{dx^n} \).

General linear ordinary differential equation of order \( n \) has form

\[
\frac{d^n f}{dx^n} + a_1(t) \frac{d^{n-1} f}{dx^{n-1}} + \cdots + a_n(t)f = g(t)
\]

Similar definition applies to partial differential equations.

In this course we focus primarily on physical systems described by linear differential equations.
Order and Linearity of Differential Equations

Example

Determine differential equation order and whether or not it’s a linear differential equation.

- \( t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + 2y = \sin t \)  
  2nd order, linear

- \( \frac{dy}{dt} + ty^2 = 0 \)  
  1st order, non-linear

- \( (1 + y^2) \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + y = e^t \)  
  2nd order, non-linear

- \( \frac{d^3 y}{dt^3} + t \frac{dy}{dt} + (\cos^2 t)y = t^3 \)  
  3rd order, linear

- \( \frac{d^4 y}{dt^4} + \frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} + \frac{dy}{dt} + y = 1 \)  
  4th order, linear

- \( \frac{d^2 y}{dt^2} + \sin(t + y) = \sin t \)  
  2nd order, non-linear
Equations of Motion in terms of Potential Energy

For $N$ particles we get $3N$ coupled partial differential equations of motion:

$$m_i \frac{d^2x_i}{dt^2} + \frac{\partial V(x_1, \ldots, x_{3N})}{\partial x_i} = 0$$

Note change in notation:

- For particle 1
  - $x_1 \rightarrow x$
  - $m_1 \rightarrow m$
  - $x_2 \rightarrow y$
  - $m_2 \rightarrow m$
  - $x_3 \rightarrow z$
  - $m_3 \rightarrow m$

- For particle 2
  - $x_4 \rightarrow x$
  - $m_4 \rightarrow m$
  - $x_5 \rightarrow y$
  - $m_5 \rightarrow m$
  - $x_6 \rightarrow z$
  - $m_6 \rightarrow m$
Equations of Motion in terms of Kinetic and Potential Energy

Since kinetic energy is

\[ K(\dot{x}_1, \ldots, \dot{x}_{3N}) = \sum_{i=1}^{3N} \frac{1}{2} m_i \dot{x}_i^2, \quad \text{where} \quad \dot{x} = \frac{dx}{dt} \]

Take partial derivative of both sides with respect to \( \dot{x}_i \)

\[ \frac{\partial K(\dot{x}_1, \ldots, \dot{x}_{3N})}{\partial \dot{x}_i} = m_i \dot{x}_i \]

Then take partial derivative with respect to time

\[ \frac{d}{dt} \left( \frac{\partial K(\dot{x}_1, \ldots, \dot{x}_{3N})}{\partial \dot{x}} \right) = m_i \frac{d\dot{x}_i}{dt} = m_i \frac{d^2 x_i}{dt^2} \]

and

\[ m_i \frac{d^2 x_i}{dt^2} + \frac{\partial V(x_1, \ldots, x_{3N})}{\partial x_i} = 0 \quad \text{becomes} \quad \frac{d}{dt} \left( \frac{\partial K(\dot{x}_1, \ldots, \dot{x}_{3N})}{\partial \dot{x}} \right) + \frac{\partial V(x_1, \ldots, x_{3N})}{\partial x_i} = 0 \]
Equations of Motion in terms of Kinetic and Potential Energy

For $N$ particles the $3N$ coupled partial differential equations of motion can be written

$$\frac{d}{dt} \left( \frac{\partial K(x_1, \ldots, x_{3N})}{\partial \dot{x}} \right) + \frac{\partial V(x_1, \ldots, x_{3N})}{\partial x_i} = 0$$

- Equations of motion can be derived from scalar functions for kinetic and potential energy.
- No force vectors are needed.
- Keep in mind that general concept of energy, and particularly those of kinetic and potential energies, did not appear until well over 100 years after Newton death.
- This kind of reformulation of Newtonian Mechanics in terms of energy functions eventually led to a more versatile approaches called Lagrangian Mechanics and Hamiltonian Mechanics (more later).
Equations of Motion in terms of Kinetic and Potential Energy

Example

Neglecting air resistance, determine trajectory of a ball initially thrown straight up in the air with an initial velocity $v_0$.

$$\frac{d}{dt} \left( \frac{\partial K(\dot{z})}{\partial \dot{z}} \right) + \frac{\partial V(z)}{\partial z} = 0$$

- Kinetic energy is $K = \frac{1}{2} m v_z^2 = \frac{1}{2} m \dot{z}^2$

- Potential energy is $V = m g_0 z$, where $g_0 = 9.80665 \text{ m/s}^2$, is 
  magnitude of acceleration due to force of gravity.

- $$\left( \frac{\partial K(\dot{z})}{\partial \dot{z}} \right) = m \ddot{z}, \quad \frac{d}{dt} \left( \frac{\partial K(\dot{z})}{\partial \dot{x}} \right) = m \ddot{x}, \quad \text{and} \quad \frac{\partial V(z)}{\partial z} = m g_0$$

- Differential equation of motion is $m \ddot{z} + m g_0 = 0$

- Solution is $z(t) = v_0 t - \frac{1}{2} g_0 t^2$ (see Problem 1-17)
Central Forces

Definition

A central force is everywhere directed towards a fixed mass or “force center”, and has the form

\[ \vec{F}(\vec{r}) = f(\vec{r}) \vec{e}_r \]

\( \vec{e}_r \) is unit vector in spherical coordinate system. Spherical coordinate unit vectors are given by

\[ \vec{e}_r = \sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z, \]
\[ \vec{e}_\theta = \cos \theta \cos \phi \hat{e}_x + \cos \theta \sin \phi \hat{e}_y - \sin \theta \hat{e}_z, \]
\[ \vec{e}_\phi = -\sin \phi \hat{e}_x + \cos \phi \hat{e}_y. \]
Conservative Central Forces

If central force is conservative then it must be spherically symmetric.

\[
\vec{F}(\vec{r}) = f(\vec{r}) \hat{e}_r
\]

Non-conservative central force:

Conservative central force:

\[
\vec{F}(\vec{r}) = f(r) \hat{e}_r
\]

3D problems with conservative central forces can often be transformed into 1D problems.
Fundamental Forces of Nature

Gravitational force: a conservative central force

Gravitational force on a mass $M$ due to mass $m$ at distance $r$ is

$$\vec{F} = G M m \frac{\vec{r}_M - \vec{r}_m}{|\vec{r}_M - \vec{r}_m|^3},$$

Newton’s law of universal gravitation

$G$ is gravitational constant, $G = 6.67408 \times 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$.

$\vec{r}_M - \vec{r}_m$ is inter-particle vector.
Fundamental Forces of Nature

Electric force: a conservative central force

Electric force on charge $Q$ due to charge $q$ which is at rest a distance $r$ away is also conservative and central force

$$\vec{F} = \frac{Qq}{4\pi \epsilon_0 |\vec{r}_Q - \vec{r}_q|^3}, \quad \text{Coulomb's law}$$

where $\epsilon_0$ is the electric constant, $\epsilon_0 = 8.854187817 \times 10^{-12}$ s$^4$A$^2$/m$^3$kg).
Fundamental Forces of Nature

Magnetic force

The magnetic force between two closed loops of wire carrying currents $I_1$ and $I_2$ is given by the path integral

$$\vec{F} = -\frac{\mu_0}{4\pi} \oint I_1 d\vec{\ell}_1 \cdot \oint I_2 d\vec{\ell}_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}. $$

$\mu_0$ is the magnetic constant

$\mu_0 = 1.256637061435917 \times 10^{-6} \text{ m} \cdot \text{kg}/(\text{s}^2 \cdot \text{A}^2)$
Fields
Michael Faraday (1791 - 1867)
Electric field

What is force that acts on charge $Q$ due to some fixed arrangement of $N$ charges, $q_\alpha$?

We can calculate superposition of forces of each $q_\alpha$ on $Q$ to obtain

$$\vec{F} = \sum_\alpha \vec{F}_\alpha = \frac{Q}{4\pi \epsilon_0} \sum_\alpha q_\alpha \frac{\vec{r} - \vec{r}_\alpha}{|\vec{r} - \vec{r}_\alpha|^3} = Q\vec{E}(\vec{r})$$

where electric field at $\vec{r}$ due to some fixed arrangement of $N$ charges, $q_\alpha$, is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi \epsilon_0} \sum_\alpha q_\alpha \frac{\vec{r} - \vec{r}_\alpha}{|\vec{r} - \vec{r}_\alpha|^3}$$
Electric field lines around single positive charge

Field lines appear closer together when field is stronger and further apart when field is weaker.
Electric field lines around two opposite sign charges

A physical dipole

Field lines always point away from positive charges and towards negative charges.
Electric field lines around two identical charges

Electric field is tangent to field line at any given point in space.
Web Apps

- 2D Electrostatic Fields
- 3D Electrostatic Fields
Magnetic field

Biot-Savart law

Force exerted on wire carrying current $I$ in magnetic field arising from all surrounding currents given by

$$
\vec{F} = \oint \vec{I} d\vec{\ell} \times \vec{B}(\vec{r}) \quad \text{where} \quad \vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi} \sum_k \oint I_k d\vec{\ell}_k \frac{\vec{r} - \vec{r}_k}{|\vec{r} - \vec{r}_k|^3}
$$
Magnetic field from a current loop

Right hand rule for direction of the magnetic field vector

If you point your right hand thumb along direction of current then your fingers will curl in direction of magnetic field.

In 1820 Ampère suggested that magnetism in matter arises from a multitude of ring currents circulating at the atomic and molecular scale.
Web Apps

- 3D Magnetostatic Fields
Lorentz force law

For charge $Q$ in motion with velocity $\vec{v}$ through region containing both electric and magnetic field in given inertial frame the force on charge is

$$\vec{F} = Q \left( \vec{E} + \vec{v} \times \vec{B} \right)$$

- Force from magnetic field changes direction of charge’s velocity but not its speed. Magnetic fields do not change particle’s kinetic energy and thus do no work on particle.
- In 1905 Albert Einstein published “On the electrodynamics of moving bodies”, showing that magnetism is a relativistic effect.
- Electric and magnetic fields transform into each other when moving between different inertial frames.

Video Links:
- Electrons in Magnetic Fields
- How Special Relativity Makes Magnets Work